<u>Queen's College 155 Anniversary Quiz</u>

1. Consider the number **39.**
   The smallest prime and the biggest prime factor of 39 are 3 and 13.
   The prime numbers between 3 and 13 are 3, 5, 7, 11, 13.
   Also, $\mathbf{39} = 3 + 5 + 7 + 11 + 13$.
   Find the next number that has each property.
   (Hint: the number is bigger than 100 and it must not be a prime number.)

1 Ans. $\underline{155} = 5 + 7 + 11 + 13 + 17 + 19 + 23 + 29 + 31$

2. The number of vertices of a right prism is greater than the number of vertices of a right pyramid by 1. If the pyramid has **155** faces, find the sum of the number of edges of the two solids.

2 Ans. Number of faces of the pyramid $= 155$

Number of **slant faces** of the pyramid $= 154$

Number of edges of all slant faces $= 154$

The base of the pyramid is a polygon of 154 sides and therefore has 154 edges.

Totol number of edges of the pyramid $= 154 + 154 = \mathbf{308}$

Total number of vertices of the pyramid $= 155$

Therefore the total number of vertices of the prism $= 156$

The base and the top of the prism is a polygon of $\frac{156}{2} = 78$ sides.

Number of edges of the of the prism $= 78 \times 3 = \mathbf{234}$

Total number of edges of the two solids $= 308 + 234 = \underline{\mathbf{542}}$

3. $15! = 15 \times 14 \times 13 \times \ldots \times 1 = 1307674368\mathbf{000}$

There are 3 trailing zeros. (Continuous number of zeros in the right side of the number.)

How many trailing zeros are there in $155!$

3 Ans. A trailing zero is formed when a multiple of 5 is multiplied with a multiple of 2. Now all we have to do is count the number of 5's and 2's in the multiplication.

Since a zero is created by $10 = 2 \times 5$ and there are more factors of 2 than 5 in $155!$, all we have to do is to count the factor 5 in the product.

Let's count the 5's first. 5, 10, 15, 20, 25 and so on, making a total $\left[\frac{155}{5}\right] = 31$, where $[x]$ denotes the largest integer smaller or equal to x.

However there are numbers 25, 50, 75, … which makes up two fives in each of them

$(25 = 5 \times 5)$, these number count up to $\left[\frac{155}{25}\right] = 6$

Similarly, we need to count up numbers which have 3 fives as factors.

Number of trailing zeros $= \left[\frac{155}{5}\right] + \left[\frac{155}{25}\right] + \left[\frac{155}{125}\right] = 31 + 6 + 1 = \underline{\mathbf{38}}$

4. Simplify $1(1!) + 2(2!) + 3(3!) + \cdots + 155(155!)$

4 Ans. $1(1!) + 2(2!) + 3(3!) + \cdots + 155(155!)$

$= [2(1!) + 3(2!) + 4(3!) + \cdots + 156(155!)]$

$\quad -[1! + 2! + 3! + \cdots + 155!]$

$= [2! + 3! + 4! + \cdots + 156!] - [1! + 2! + 3! + \cdots + 155!]$

$= \underline{\mathbf{156! - 1}}$

5. Find the **centre number** of the 155^{th} row the Kordemsky's triangular array. Find also the sum of all numbers in this row.

Kordemsky's infinite triangular array

1
2 3 4
5 6 7 8 9
10 11 12 13 14 15 16
17 18 19 20 21 22 23 24 25
26 27 28 29 30 31 32 33 34 35 36
37 38 39 40 41 42 43 44 45 46 47 48 49
50 51 52 53 54 55 56 57 58 59 60 61 62 63 64
65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80 81
82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100

5 Ans. The last number of the n^{th} row $= n^2$

The centre number of the n^{th} row $= n^2 - n + 1$

The centre number of the 155^{th} row $= 155^2 - 155 + 1 = \underline{\mathbf{23871}}$

There are $(2n - 1)$ numbers in the r^{th} row.

The first number in the n^{th} row $= n^2 - (2n - 1) + 1 = n^2 - 2n + 2$

The first number in the 155^{th} row $= 155^2 - 2(155) + 2 = 23717$

Sum of all numbers in the 155^{th} row $= \frac{2n-1}{2}(a + l) = \frac{2(155)-1}{2}(23717 + 155^2) = \underline{\mathbf{7376139}}$

6. If $f(x^2-313x)=(x-155)(x-156)(x-157)(x-158)$, find $f(x-155^2)$.

6 Ans. $f(x^2-313x)=(x-155)(x-156)(x-157)(x-158)$

$=[(x-155)(x-158)][(x-156)(x-157)]$

$=[x^2-313\,x+24490][x^2-313\,x+24492]$

Hence, $f(x)=(x+24490)(x+24492)$

Replace x by $x-155^2$,

$f(x-155^2)=(x-155^2+24490)(x-155^2+24492)$

$=\underline{\mathbf{(x+465)(x+467)}}$

7. The sequence $x_1,x_2,x_3,\dots,x_{155},x_{156},\dots$ satisfies :

$$x_1=\frac{1}{2},\quad x_{k+1}={x_k}^2+x_k \text{ where } k=1,2,\dots,155,\dots.$$

Find the integral part (that is, excluding the decimal part) of the sum

$$\frac{1}{x_1+1}+\frac{1}{x_2+1}+\cdots+\frac{1}{x_{155}+1}.\qquad \left(\text{Hint: } \frac{1}{x_1}-\frac{1}{x_1+1}.\right)$$

7 Ans. We have $\frac{1}{x_1}-\frac{1}{x_1+1}=\frac{1}{x_1(x_1+1)}=\frac{1}{x_2}$

Similarly, $\frac{1}{x_2}-\frac{1}{x_2+1}=\frac{1}{x_3}$

Work up to $\frac{1}{x_{154}}-\frac{1}{x_{155}+1}=\frac{1}{x_{156}}$

Add all these identities up, we have $\frac{1}{x_1+1}+\frac{1}{x_2+1}+\cdots+\frac{1}{x_{155}+1}=\frac{1}{x_1}-\frac{1}{x_{156}}=2-\frac{1}{x_{156}}$

Since $x_1=\frac{1}{2}, x_2=\frac{1}{4}+\frac{1}{2}=\frac{3}{4}, x_3=\left(\frac{3}{4}\right)^2+\frac{3}{4}=\frac{21}{16}>1$, we have

$x_1<x_2<x_3<\cdots<x_{156}$ (More serious reader may use mathematical induction.)

Then $1=2-1<2-\frac{1}{x_{156}}<2-0=2$

The integral part of $\frac{1}{x_1+1}+\frac{1}{x_2+1}+\cdots+\frac{1}{x_{155}+1}$ is $\underline{\underline{\mathbf{1}}}$.

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